Math 246C Lecture 22 Notes

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1 Regularization of Subharmonic Functions and L^2 Estimates for the $\overline{\partial}$ Operator

1.1 Regularization of subharmonic functions

Let $u \in \mathrm{SH}(\Omega)$ be $u \not\equiv -\infty$. Let $0 \leq \varphi \in C_0^\infty(\mathbb{C})$ be such that $\varphi = 0$ for $|z| \geq 1$, φ is radially symmetric, and $\int \varphi = 1$. Define

$$u_\varepsilon = u * \varphi_\varepsilon = \int u(z-\zeta)\varphi_\varepsilon(\zeta)\, L(d\zeta), \qquad \varphi_\varepsilon(z) = \frac{1}{\varepsilon^2}\varphi\left(\frac{z}{\varepsilon}\right),$$

and let $\Omega_\varepsilon = \{z \in \Omega : \mathrm{dist}(z, \Omega^c) > \varepsilon\}$,

Proposition 1.1. *$u_\varepsilon \in (C^\infty \cap \mathrm{SH})(\Omega_\varepsilon)$, and $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$.*

Proof. We have already shown the first statement, and we have shown that $u_\varepsilon \geq 0$ for all $\varepsilon > 0$.

We want to check that $u_\varepsilon \downarrow u$ as $\varepsilon \downarrow 0$. As φ is radially symmetric, we have

$$u_\varepsilon(z) = \int \varphi(r) r \underbrace{\left(\int_0^{2\pi} u(z + \varepsilon r e^{it})\, dt\right)}_{\text{increasing with } \varepsilon} dr.$$

We get that $\lim_{\varepsilon \to 0} u_\varepsilon \in \mathrm{SH}(\Omega)$ and is $\geq u$. On the other hand, by Fatou's lemma,

$$\limsup_{\varepsilon \to 0} \int u(z + \varepsilon\zeta)\varphi(\zeta)\, L(d\zeta) \leq \int \limsup_{\varepsilon \to 0} u(z+\varepsilon\zeta)\varphi(\zeta)\, L(d\zeta) \leq u(z)$$

by the upper semicontinuity of u. So $u_\varepsilon \downarrow u$. $\square$

Remark 1.1. Regularization arguments show the following: if $u \in \mathrm{SH}(\Omega)$, where $u \not\equiv -\infty$ and Ω is connected, then

$$\int u \Delta\varphi\, L(ds) \geq 0 \qquad \forall 0 \leq \varphi \in C_0^\infty(\Omega).$$

Conversely, assume that $U \in L^1_{\text{loc}}(\Omega)$ such that

$$\int U \Delta\varphi \, L(d\zeta) \geq 0.$$

Then there exists a unique $u \in \mathrm{SH}(\Omega)$ such that $u = U$ a.e.

1.2 L^2 estimates for the $\overline{\partial}$ operator

Let $\Omega \subseteq \mathbb{C}$ be open. Consider the Cauchy-Riemann equation

$$\frac{\partial u}{\partial \overline{z}} = f.$$

Recall thatif $f \in C^\infty(\Omega)$, there exists some $u \in C^\infty(\Omega)$ solving this equation. We want to solve the equation with $f \in L^2_{\text{loc}}(\Omega)$ and get *estimates* for the solution.

Definition 1.1. Let $f \in L^2_{\text{loc}}(\Omega)$. We say that $u \in L^2_{\text{loc}}$ is a **solution in the weak sense** of the Cauchy-Riemann equation if for all $\eta \in C_0^\infty(\Omega)$,

$$-\int u \partial_{\overline{z}} \beta \, L(dz) = \int f \beta \, L(dz).$$

Theorem 1.1 (Hörmander[1])**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let $\varphi \in C^\infty(\Omega)$ be strictly subharmonic: $\Delta\varphi > 0$ in Ω. Then, for any $f \in L^2_{\text{loc}}(\Omega)$ such that*

$$\int \frac{|f|^2}{\varphi''_{z,\overline{z}}} e^{-\varphi} \, L(dz) < \infty,$$

there exists a weak solution $u \in L^2_{\text{loc}}(\Omega)$ to $\frac{\partial u}{\partial \overline{z}} = f$ such that

$$\int_\Omega |u|^2 e^{-\varphi} \, L(dz) \leq \int_\Omega \frac{|f|^2}{\varphi''_{z,\overline{z}}} e^{-\varphi} \, L(dz).$$

Proof. We shall work in the Hilbert space

$$L^2_\varphi = L^2(\Omega, e^{-\varphi}) = \left\{ f : \Omega \to \mathbb{C} \text{ measurable} \mid \|f\|_{L^2_\varphi} := \int |f| e^{-\varphi} \, L(dz) < \infty \right\}.$$

Consider the linear operator $T : L^2_\varphi \to L^2_\varphi$ given by $Tu = \frac{\partial u}{\partial \overline{z}}$ equipped with the domain

$$D(T) = \left\{ u \in L^2_\varphi : \exists f \in L^2_\varphi \text{ s.t. } f = \frac{\partial u}{\partial \overline{z}} \text{ weakly: } -\int u \partial_{\overline{z}} \beta = \int f \beta \; \forall \beta \in C_0^\infty(\Omega) \right\}.$$

[1] This result, unlike the other results we have been proving, is fairly recent. It was proven in 1965.

Then $D(T)$ is dense in L^2_φ, and $Tu = f$.

We have the adjoint $T^* =: \overline{\partial}^*_\varphi$ of T:

$$\langle \overline{\partial}, \beta \rangle_{L^2_\varphi} = \langle u, \overline{\partial}^*_\varphi \beta \rangle_{L^2_\varphi} \qquad \forall u \in D(T), \beta \in C_0^\infty(\Omega).$$

Compute $\overline{\partial}^*_\varphi$:

$$\langle \partial \overline{u}, \beta \rangle_{L^2_\varphi} = \int \overline{\partial} u \underbrace{\overline{\beta} e^{-\varphi}}_{\in C_0^\infty} L(dz) = -\int u \partial_{\overline{z}} (\overline{\beta} e^{-\varphi}) \, L(dz) = \int u \overline{\overline{\partial}^*_\varphi \beta} e^{-\varphi} \, L(dz).$$

We get that

$$\overline{\partial}^*_\varphi \beta = -e^\varphi \partial_z (\beta e^{-\varphi}) = -\partial_z \beta + (\partial_z \varphi) \beta.$$

The idea is that to get a solvability result for $\overline{\partial}$ acting on L^2_φ, we need an a priori estimate for $\overline{\partial}^*_\varphi$. □

Before we continue with the proof, we need the following proposition:

Proposition 1.2. *Let $f \in L^2_{\mathrm{loc}}(\Omega)$, and let $C > 0$ be constant. Then there exists a $u \in L^2_{\mathrm{loc}}(\Omega)$ such that $\overline{\partial} u = f$ and $\int |u|^2 e^{-\varphi} \, L(dz) \le C$ if and only if*

$$\left| \int f \overline{\beta} e^{-\varphi} \, L(dz) \right| \le C \int |\overline{\partial}^*_\varphi \beta|^2 e^{-\varphi} \, L(dz) \qquad \forall \beta \in C_0^\infty(\Omega).$$

Proof. ($\implies$): We have by Cauchy-Schwarz that

$$\left| \int f \overline{\beta} e^{-\varphi} \, L(dz) \right| = \left| \int \overline{\partial} u \overline{\beta} e^{-\varphi} \, L(dz) \right| = |\langle u, \overline{\partial}^*_\varphi \beta \rangle_{L^2_\varphi}| \le C^{1/2} \|\overline{\partial}^*_\varphi \beta\|_{L^2_\varphi}.$$

($\impliedby$): Assume that the bound holds. The linear functional

$$F(\overline{\partial}^*_\varphi \beta) = \overline{\int f \overline{\beta} e^{-\varphi} \, L(dz)}.$$

is well-defined on $\overline{\partial}^*_\varphi C_0^\infty(\Omega) \subseteq L^2_\varphi$, and $|F(\overline{\partial}^*_\varphi \beta)| \le C^{1/2} \|\overline{\partial}^*_\varphi \beta\|_{L^2_\varphi}$. So its norm is $\le C^{1/2}$. By the Hahn-Banach theorem, F extends to all of L^2_φ. So there is a $u \in L^2_\varphi$ representing the linear functional F. □